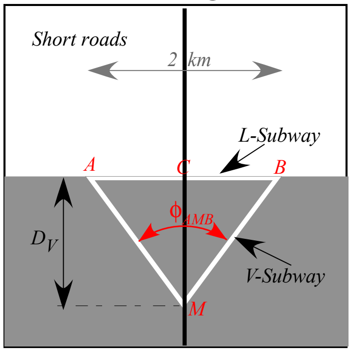
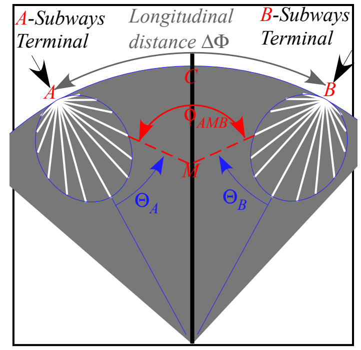
Oct. 2 2019 - Assignment 6 - due Wed Oct. 09 - Mainly Chapters 9 , 11, and 12. Lect. 11 Name____________





Geometric Optimization Exercises

Fig. 1 Local friction-free subways on Sophomore Physics-Earth (SPE). Fig. 2 Global friction-free subways on SPE.

1. a. Friction-free local subway path *ACB* in Fig. 1 is laser-beam straight and normal to line thru Earth center only at turning point *C*. How long does it take for subway car starting with $v(0)=0$ at point *A* to arrive at point *B*?

b. Assume *constant* gravity $g=9.8m/s^2$ for friction-free local subway path *AMB* in Fig. 1 where turning vertex *M* is negotiated ideally with no loss of energy (or life!). Derive depth D_V of point *M* and angle ϕ_{AMB} that gives the quickest trip from *A* to *B* ($AC=1km=CB$) thru *M* and derive the *AMB* time of travel τ_{AMB}=______ *min*.

c. Cycloid (Lect. 11 p.33-42) gives *absolute* minimum τ_{AB}. Derive its (x,y) formula and plot using attached cycloid graph paper. Over it plot the optimal V-shaped path *AMB* of 1b to help compare the two.

d. Use Lagrange equation of circle angle ϕ to find cycloid travel time τ_{AB}=______*min* and compare it to τ_{AMB}.

2. Assume *Isotropic Harmonic Oscillator* IHO gravity in Fig. 2 with acceleration $g=9.8m/s^2$ at surface dropping to $g=0$ at Earth-center $C_\oplus$ (bottom-center of Fig.2). The objective (as in Ex. 1) is to find path *AMB* and angle ϕ_{AMB} having least time of travel τ_{AMB} between Terminal points *A* and *B* separated by great circle longitude angle $\Delta\Phi_{AB}$.

Before solving main objective consider some alternative routes whose travel times should be easy to derive.

a. Direct straight line route from *A* to *B*: $\tau_{A\ direct\ toB}$=______ *min*. Relate to SPE half-orbit period $\tau_\odot/2$

b. Straight line segment routes *A* to *C* then *C* to *B*: $\tau_{AtoCtoB}$=______ *min*. " "

c. Direct *A* to Earth-center $C_\oplus$ then $C_\oplus$ back up to *B*: $\tau_{AtoC_\oplus toB}$ =______ *min*. " "

3. To solve main objective of Ex.2, imagine subway cars from terminal *A* or *B* leave their terminal at time $t_0=0$ and fall along straight tunnels (white lines) to positions at later time t_1 indicated by points on blue oval in Fig. 2. Ovals *A* and *B* expand equally until a touch-time t_T when they are tangent to each other and to vertical line *CM*.

a. That touch-time t_T is related to total minimum travel time τ_{AMB}. How? (Recall τ_{AMB} for Ex. 1.)

b. Derive polar $r(\Theta,t)$ equation for "ovals" relative to Terminal origin. What Thales geometric form is it?

c. Relate optimizing angle ϕ_{AMB} to angle $\Delta\Phi_{AB}$ of longitudinal *A* to *B* separation. Plot geometry for $\Delta\Phi=\frac{\pi}{2}$.
It helps to define a slope angle α between optimal subway path and terminal vertical radial line.

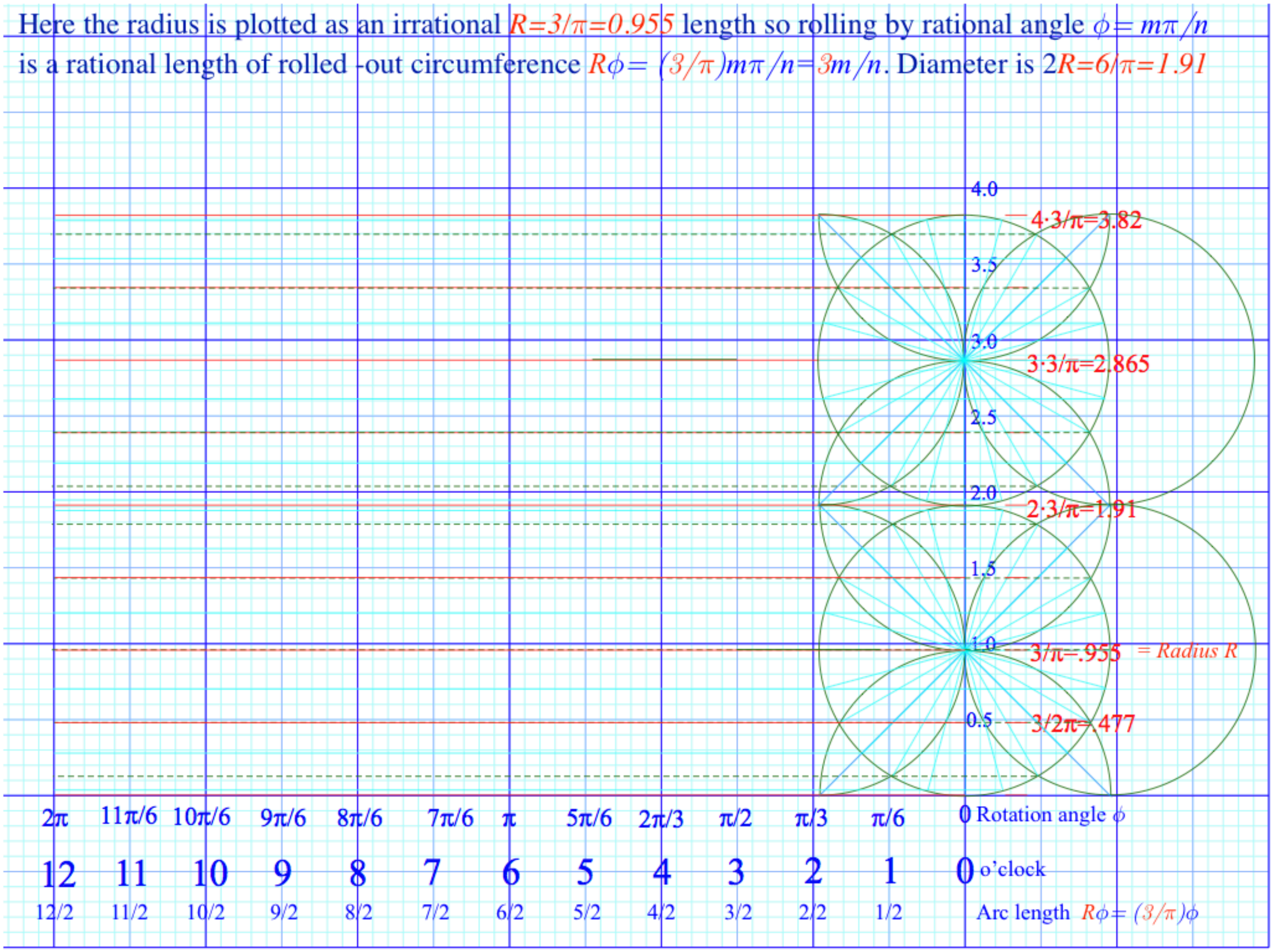
d. A terminal-launched SPE circular orbit serves as a clock hand that quantifies growing "oval" size.
Include this in your geometric plot to quantify optimal travel time τ_{AMB} and plot car positions vs. t.
Show car positions at fractions $\frac{1}{12},\frac{2}{12},\frac{3}{12}$,...of half-orbit period $\tau_\odot/2$ to help relate to travel time τ_{AMB}.

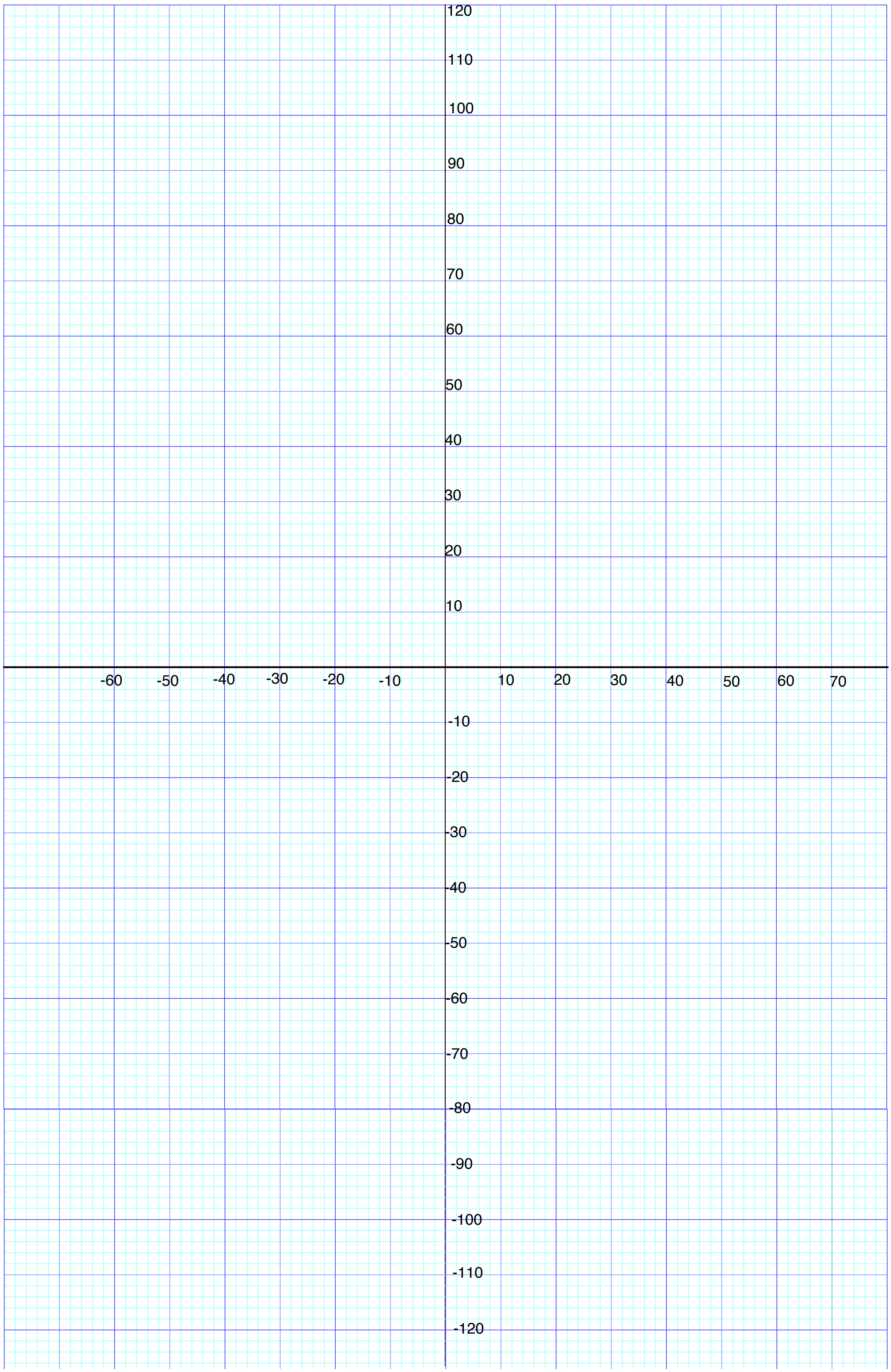
e. Verify geometry c and d in extreme cases of distance that is small ($\Delta\Phi\ll\frac{\pi}{2}$ like Ex.1) or large ($\Delta\Phi=\pi$).

For Exercises 1c and 1d:

Huygen's cycloidal-pendulum construction graph (See Lect. 11p.33-42).

Motion may be calculated using Lagrange's 1D equation using single angle variable ϕ and angular velocity $\dot{\phi}$.
In *mks* units the circumference $2\pi R$ of the circle (and width *ACB* of cycloid) should be 2*km* as shown in Fig. 1.

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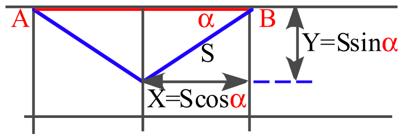
Exercise Set 6 Solutions 1a. Car falls "down" *A* to *C* in 21 min. and back "up" to *B* in *21 min*. Total time: *42 min*.
Exercise 1b The simplest local (constant gravity) travel time is greatly reduced by a "V" shaped path between A and B that drops by angle α below the line connecting *A* to *B*. For nearby points, the connecting line is very close to the horizon. Not so for more global excursions.



Local uniform g approximation

For local V-path let A and B be L=*2X* apart and connected by an isosceles "V" of slant length *S*. Uniform gravity potential *V=mgy* is zero at the surface as is assumed the initial velocity *v(t=0)* and KE*(t=0)*. Then total energy *E* will be conserved at zero value.

$$0 = E = KE + PE = \tfrac{1}{2} mv^2 + mgy = \tfrac{1}{2} m\left(\frac{ds}{dt}\right)^2 - mgs\sin\alpha \quad \text{where: } y = -s\sin\alpha \quad \text{and: } v = \frac{ds}{dt} = \sqrt{2gs\sin\alpha}$$

The travel time is then a simple integral. Consider just the first (or last) *half* travel time.

$$T = \int_0^T dt = \int_0^S \frac{ds}{v} = \int_0^S \frac{ds}{\sqrt{2gs\sin\alpha}} = \frac{1}{\sqrt{2g\sin\alpha}}\int_0^S s^{-1/2} ds = \frac{S^{1/2}}{\frac{1}{2}\sqrt{2g\sin\alpha}} = \sqrt{\frac{2S}{g\sin\alpha}} \quad \text{where: } X = S\cos\alpha$$

$$time: T_{AtoX} = \sqrt{\frac{2X}{g\sin\alpha\cos\alpha}} \qquad frequency: \upsilon_{AtoBtoA} = \frac{1}{4T_{AtoX}} = \frac{1}{4}\sqrt{\frac{g\sin\alpha\cos\alpha}{2X}} = \frac{1}{8}\sqrt{\frac{g}{X}}\sqrt{\sin 2\alpha}$$

Finding α-min-max for either υ or *T* involves zeroing its α-derivative.

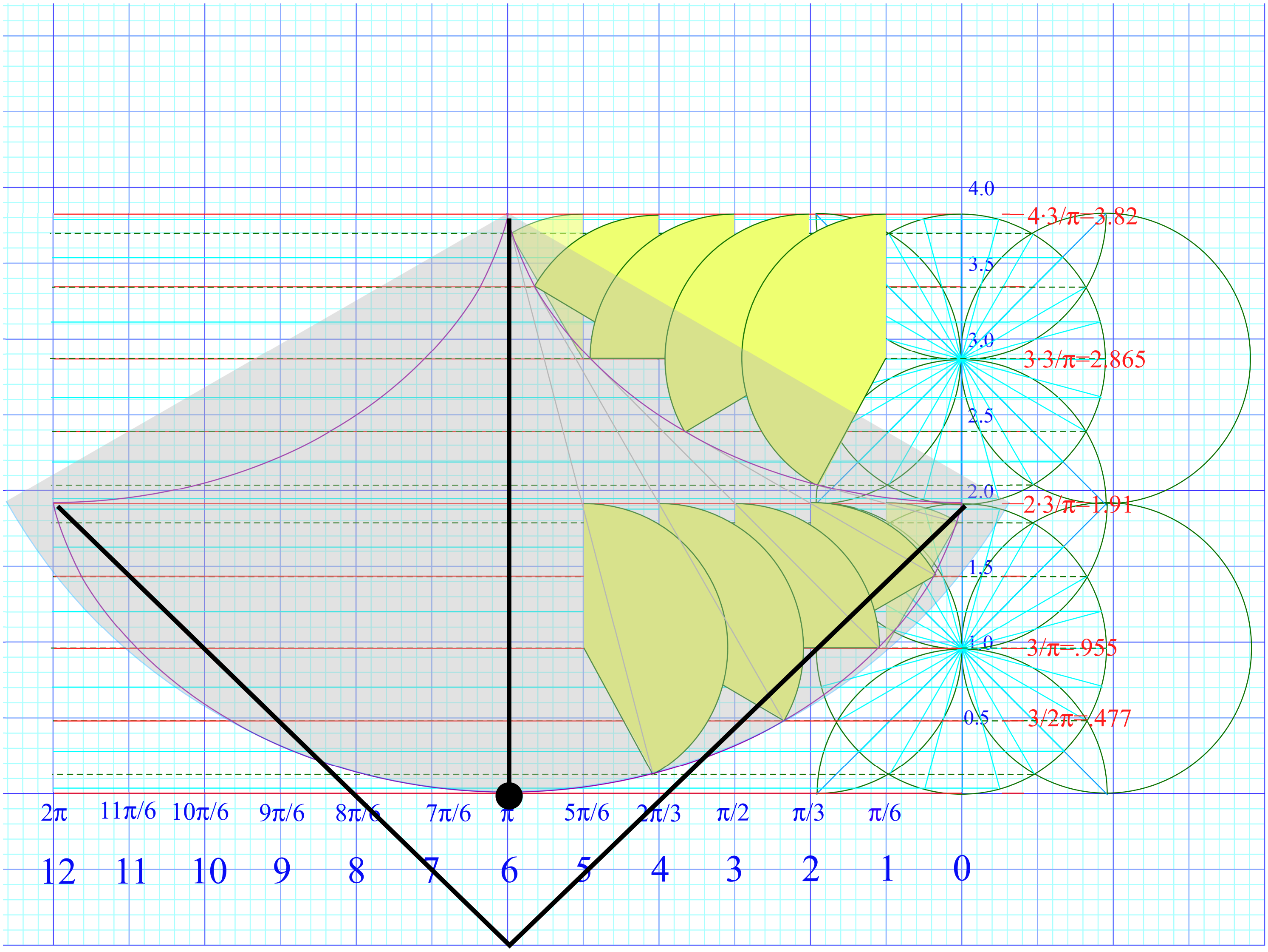
$$\frac{d\upsilon}{d\alpha} = 0 = \frac{1}{2}\sqrt{\frac{g}{X}}\frac{d}{d\alpha}\sqrt{\sin 2\alpha} = \frac{1}{2}\sqrt{\frac{g}{X}}\frac{\cos 2\alpha}{\sqrt{\sin 2\alpha}} \Rightarrow \cos 2\alpha = 0 \Rightarrow \alpha = \pm\frac{\pi}{4}$$

So angle α=45° minimizes the half-travel time as well as twice that, the whole A-to-X-to-B time.

$$T_{AtoXtoB}(1way) = 2\sqrt{\frac{2X}{g(1/\sqrt{2})^2}} = 4\sqrt{\frac{X}{g}} = 2\sqrt{2}\sqrt{\frac{L}{g}} = 2\sqrt{2}\sqrt{\frac{2\cdot 10^3 m}{9.8m/s^2}} = 40.4sec \text{ for: } L = AtoB \text{ distance=2X=2}km$$

An *A-to-B* distance of *L=2km* on Earth is thus traversed in a little over *40 seconds*. A longitude Δϕ=*1°* gives *L=111km* and *T=301 seconds* or nearly *5 min*. (Both beat the 42-minute half-circum-period by a lot!.)

$-\pi R$_________________________________ 0 __________________________________ $+\pi R$

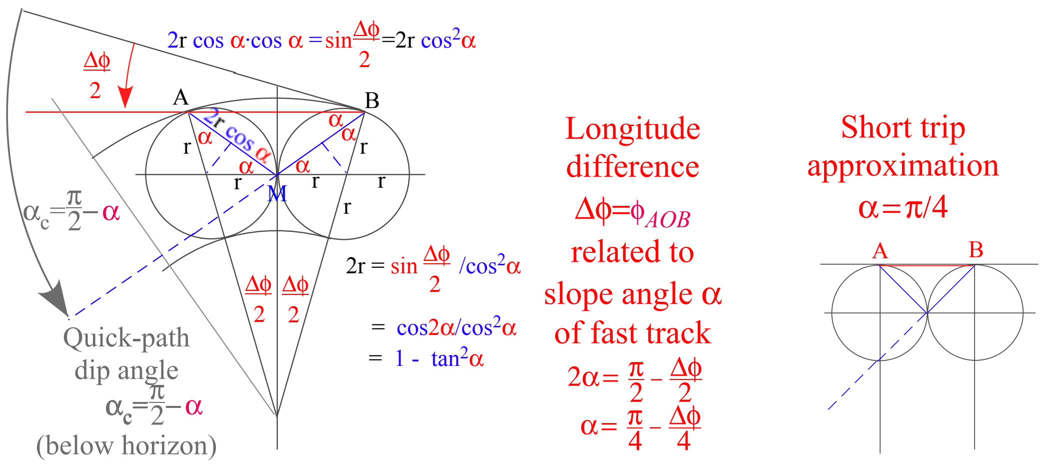


Pendulum of length 4R has frequency: $\upsilon = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{4R}}$ where: $\pi R = 1km = 10^3 m$

Pendulum of length 4R has period: $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4R}{g}}$

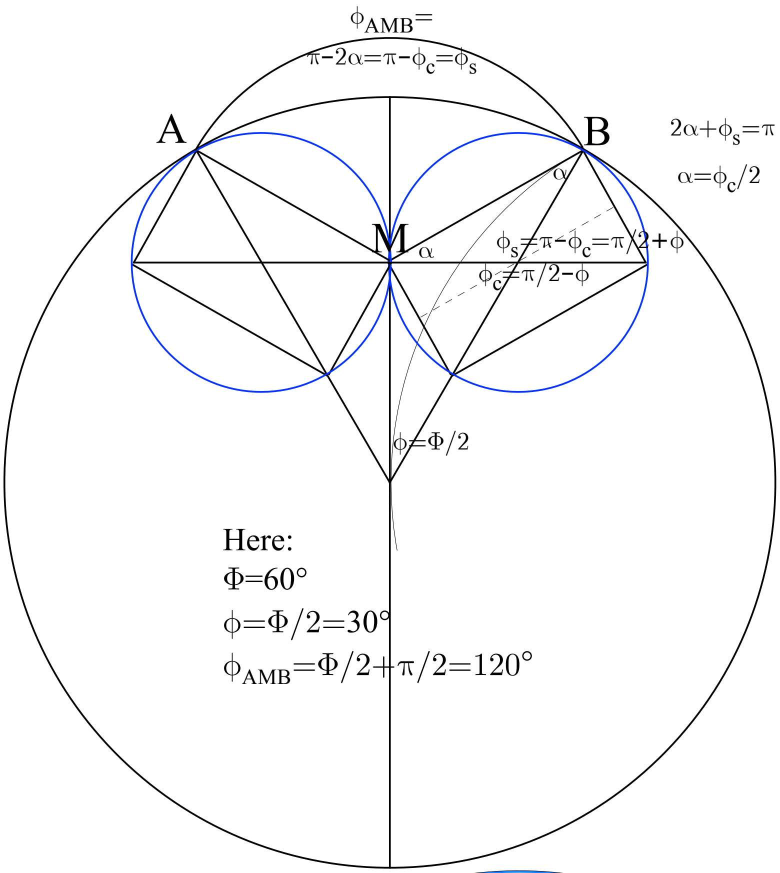
Half-period: $\frac{\tau}{2} = \pi\sqrt{\frac{4R}{g}} = \pi\sqrt{\frac{4\cdot 10^3 m}{g\cdot\pi}} = \sqrt{\frac{4\pi\cdot 10^3 m}{9.8 m\cdot s^{-2}}} = 35.81\,seconds$

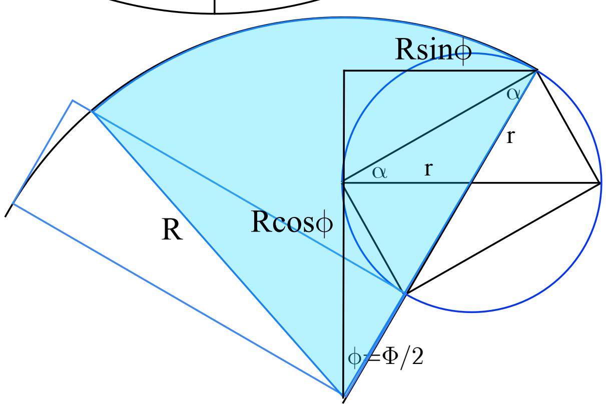
This a little quicker that the 40.4 seconds travel time on the 45° V-path.



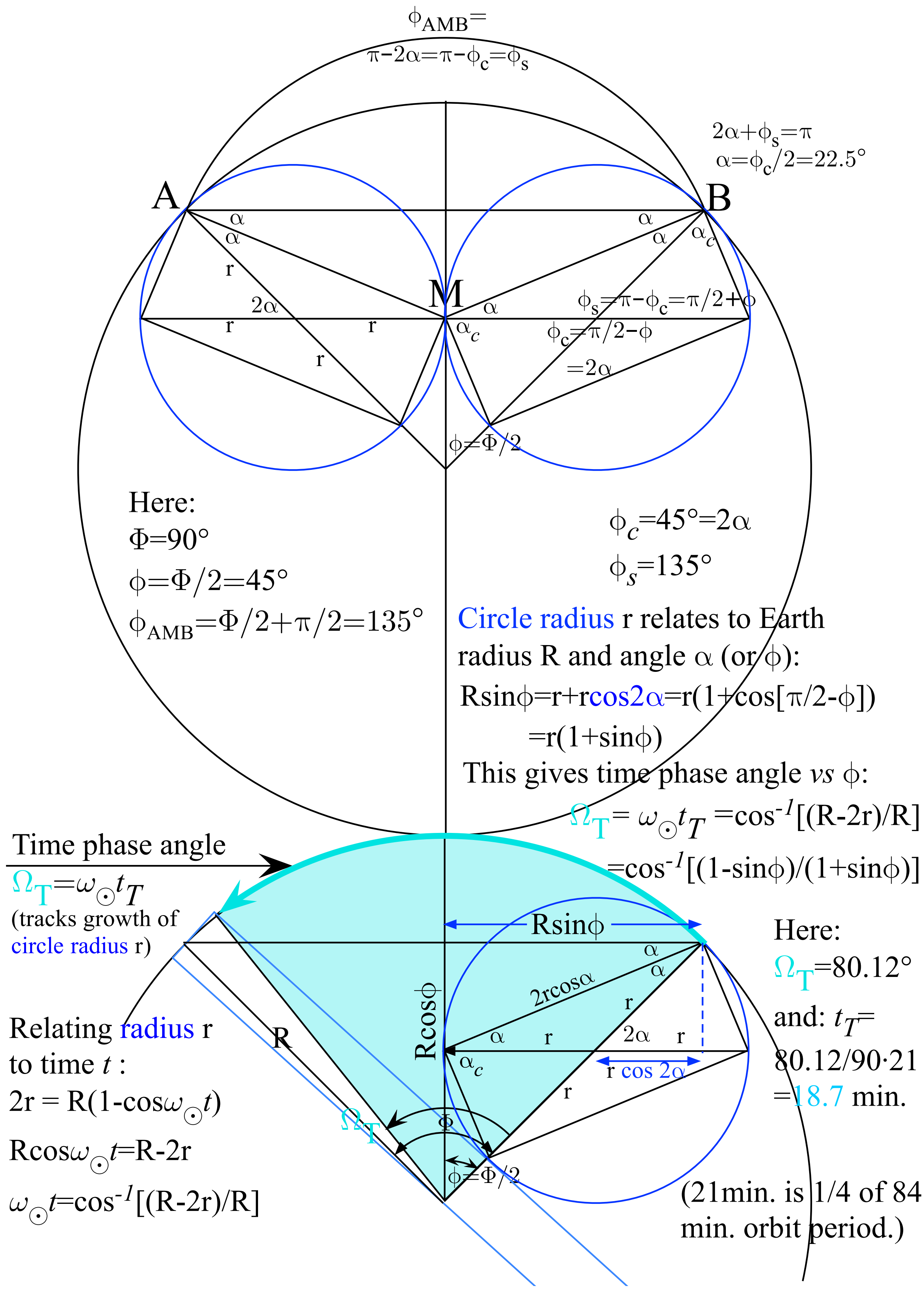
The more difficult problem of deep-V-tunnel global travel is solved similarly, but a geometric solution sketched above and below is quick (once you see the trick!). The trick is to imagine a pencil of competing tunnels going out from both point A and point B so that the trial runs from two expanding circles that finally touch on a tangent that bisects the A-to-B longitude angle $\phi_{AOB}=\Delta\phi$. We find the angle $\alpha=\pi/4-\Delta\phi/4$ between shortest path and *quickest* path. It approaches $\alpha=45°$ in the local limit $\Delta\phi \to 0$. The *AMB* vertex angle is $\phi_{AMB}=\pi/2+\phi_{AOB}/2$ and approaches a local *90°* limit. Half the *AMB* vertex angle is $\phi_{AMB}/2=\pi/2-\alpha=\alpha_C$ (compliment of α) that is also horizon dip angle between the horizon and the quickest path. For short trips: $\alpha=\alpha_C=45°$. For longer trips: $\alpha<45°$ and $\alpha_C>45°$.

Each circle diameter $D=2r$ (in units of Earth radius $R_\oplus$) expands as $D=1-\cos\theta$ where $\theta=\omega t$ is the circular orbit angle subtended by projecting the diameter point to the Earth circle. Travel time T is proportional to angle θ with $\theta=\pi$ corresponding to 42 minutes of a half-circle orbit and $\theta=\pi/2$ to 21 min. (Going half-way between A and B by the straight tunnel takes 21 minutes.)

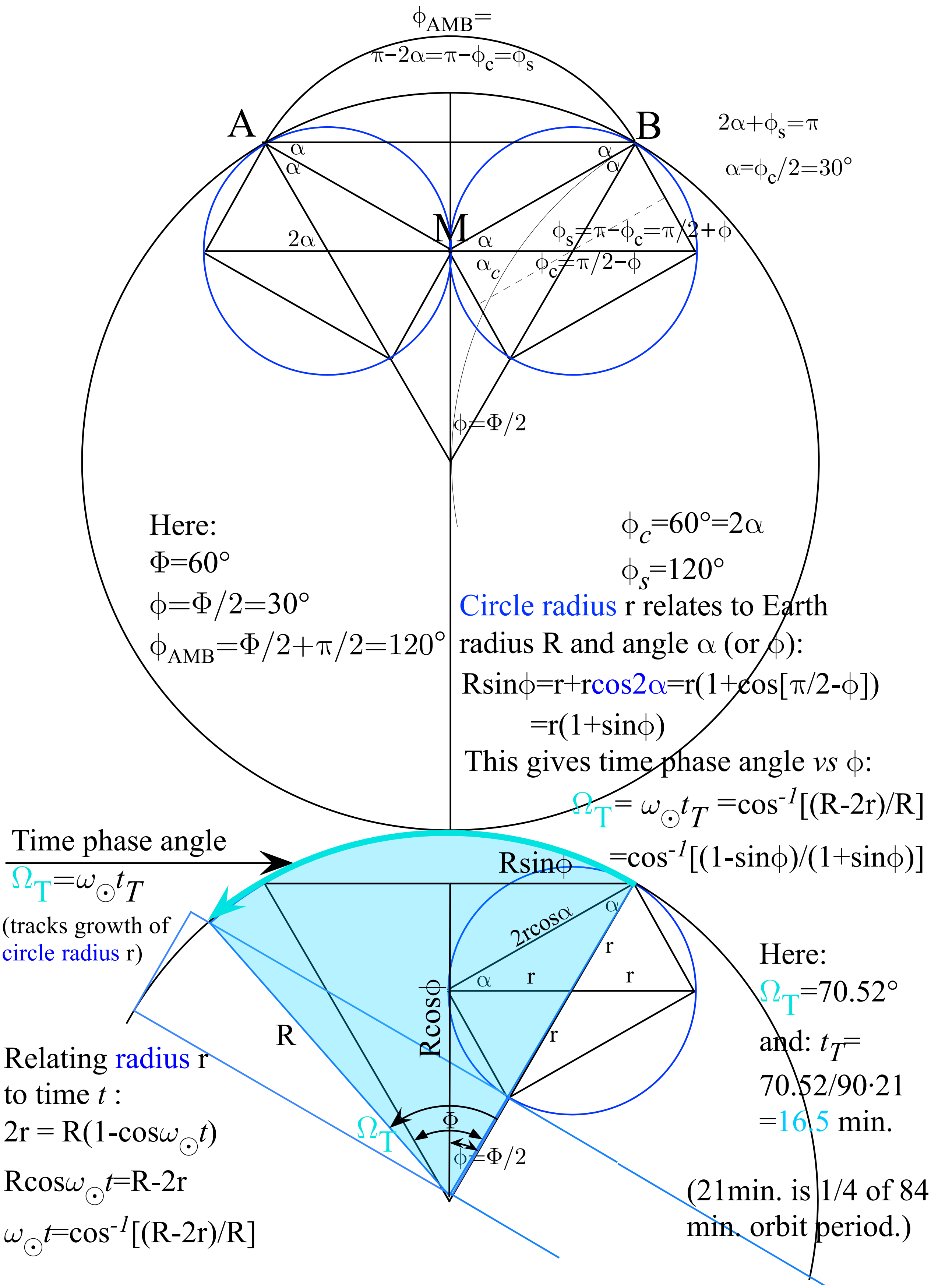




The following is a general solution with the $\phi=\Phi/2=45°$ case with numerical timing estimates.



The following is a general solution with the $\phi=\Phi/2=30°$ case given numerically.


ϕAMB=
π−2α=π−ϕc=ϕs
A
B
2α+ϕs=π
α=ϕc/2=30°
α
α
α
α
M
2α
α
αc
ϕs=π−ϕc=π/2+ϕ
ϕc=π/2−ϕ
ϕ=Φ/2
Here:
Φ=60°
ϕ=Φ/2=30°
ϕAMB=Φ/2+π/2=120°
ϕc=60°=2α
ϕs=120°
Circle radius r relates to Earth
radius R and angle α (or ϕ):
Rsinϕ=r+rcos2α=r(1+cos[π/2-ϕ])
=r(1+sinϕ)
This gives time phase angle vs ϕ:
ΩT= ω⊙tT =cos-1[(R-2r)/R]
=cos-1[(1-sinϕ)/(1+sinϕ)]
Time phase angle
ΩT=ω⊙tT
(tracks growth of
circle radius r)
Rsinϕ
2rcosα
α
r
α
r
r
Rcosϕ
R
r
Relating radius r
to time t :
2r = R(1-cosω⊙t)
Rcosω⊙t=R-2r
ω⊙t=cos-1[(R-2r)/R]
ΩT
Φ
ϕ=Φ/2
Here:
ΩT=70.52°
and: tT=
70.52/90·21
=16.5 min.
(21min. is 1/4 of 84
min. orbit period.)