Assignment 11 - **PHYS 5103**-11/13/19-Due Wed. Nov. 20 CMwBang! Ch 4.1 thru Ch.4.4. and Lectures 21-23

Ex.1(a) Do ABCD expansion of Hamiltonian $\mathbf{H} = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \vec{\mathbf{S}}$ using p. 54 of Lect. 22 for $\mathbf{H} = \frac{1}{5}\begin{pmatrix} 34 & -12 \\ -12 & 41 \end{pmatrix} = \sqrt{\mathbf{K}}$

(A square-root in last part of Assignment 10.) Evaluate crank vector $\vec{\mathbf{\Omega}}=(\Omega_A,\Omega_B,\Omega_C)$ and overall frequency Ω_0 =_____. Evaluate beat or splitting frequency $\frac{1}{2}|\Omega|$=_____ and plot eigen-frequency levels. (See p.50-54 of Lect.23)

(b) Sketch a 3D ABC-space showing the crank vector $\vec{\Omega}=(\Omega_A,\Omega_B,\Omega_C)$ and the angle it makes with the ***A***-axis.

(c) Show how a rotation $|\Theta| = |\Omega| t = \pi$ of a spin-vector **S** starting on the ***A***-axis would end up.

(d) Do ABCD expansion of 2D Hooke spring matrix **K**. Compare resulting crank vector $\vec{\Omega}$ and eigenvectors of **K** with those of **H**.

(e) Given initial conditions (*X(0)=1,Y(0)=0,*$\mathbf{V}_0$*=0*), derive and plot the resulting (*Tschebycheff*) path in the XY-plane .

Ex.2(a) Do ABCD expansion of Hamiltonian $\mathbf{H} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} = \sqrt{\mathbf{K}}$ and plot eigen-frequency levels as in **Ex.1.**

(b) Sketch a 3D ABC-space as in **Ex.1**.

(c) Show rotations as in **Ex.1**.

(d) Do ABCD expansion of 2D Hooke spring matrix **K** as in **Ex.1**.

(e) Given initial conditions (*X(0)=1,Y(0)=0,*$\mathbf{V}_0$*=0*), derive and plot a resulting (*Tschebycheff*) path in the XY-plane .

Ex.3 Derivation in Lect. 23 (p.47-65) of eigenvectors equates spin-vector **S**(α,β,·) to crank-vector ±**Ω[**φ,ϑ,·**]**.
Try using the Projector operator method introduced in Lect. 21 p. 40-44. Are results the same? Compare ease of use.

Ex.4 The clearest example of 2-state or spin ½ resonance behavior are A, B or C type of evolution in which the spin vector moves from axis to axis as demonstrated by animations accessed on pages 71 thru 88.
(a) Uncoupled oscillation of the AD type (p.71). Let Hamiltonian have A=1.5 and D=0.5 with B=0=C with initial oscillator variables $x_1(0)=1=x_2(0)$ and $p_1(0)=0=p_2(0)$. Describe initial spin vector (p.80 Lect.22) and its evolution.
(b) Balanced coupled oscillation of the B type (p.76). Explain why phase lag is always π/2 when initial position and velocity of just one oscillator is zero. (Helpful hints on p. 94-97 or p.87 of Lect. 21.)
(c) Coriolis coupled oscillation of the C type (p.87). How does this resemble motion of a Foucault pendulum?

Ex.5 For any of the apps discussed in **Ex.4** you may access the control panel and reset the A, B, C, an D parameters that determine the crank vector **Ω** and overall frequency Ω_0. See if you can set these and the spin vector S so S goes from the A-axis to the B-axis to the C-axis and back thru the A-axis. Make Ω_0 at least 40 times larger than the other A, B, and C components of **Ω**. Show a screen clip picture of the resulting evolution.

Assignment 11. Solution to **Ex.1(a)** Spin expansion for driver matrices:

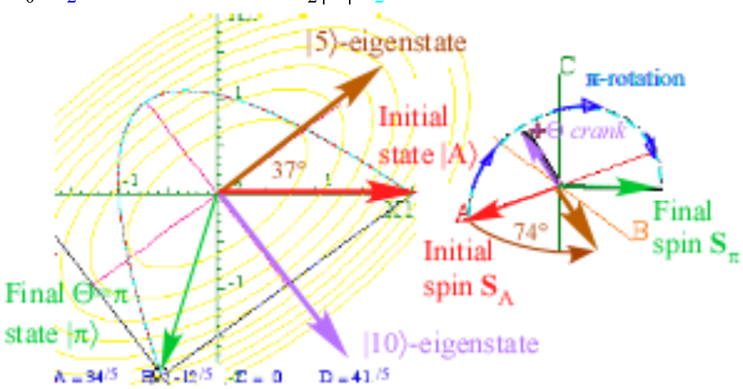
$$\mathbf{H}=\tfrac{1}{5}\begin{pmatrix}34 & -12\\ -12 & 41\end{pmatrix}=\frac{15}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{-7}{5}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}+\frac{-24}{5}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$$

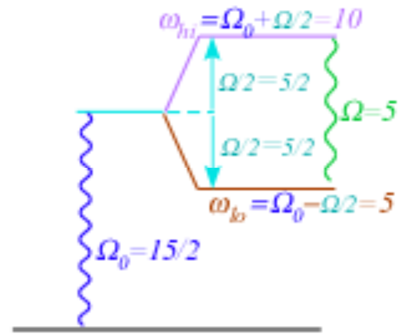
$$=\tfrac{15}{2}\mathbf{1}+\tfrac{-7}{5}\mathbf{S}_A+\tfrac{-24}{5}\mathbf{S}_B=\tfrac{15}{2}\mathbf{1}+\tfrac{-7}{10}\boldsymbol{\sigma}_A+\tfrac{-24}{10}\boldsymbol{\sigma}_B$$

$$=\tfrac{15}{2}\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B\ \therefore\left[\vec{\boldsymbol{\Omega}}=(\Omega_A,\Omega_B,\Omega_C)=(\tfrac{-7}{5},\tfrac{-24}{5},0)\right]$$

$$\mathbf{H}^2=\mathbf{K}=\begin{pmatrix}52 & -36\\ -36 & 73\end{pmatrix}=\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}$$

$\Omega_0=\frac{15}{2}$ $\qquad\frac{1}{2}|\boldsymbol{\Omega}|=\frac{5}{2}$





Solution to **Ex.1(a)-(e)** *views of coupled oscillation*

ABCD decomposition shows **H** and **K** share unit crank vector $\hat{\boldsymbol{\Omega}}=(\frac{7}{25},\frac{24}{25},0)$ and e-vectors $\begin{pmatrix}4/5\\ 3/5\end{pmatrix},\begin{pmatrix}3/5\\ -4/5\end{pmatrix}$.

$$\mathbf{H}=\tfrac{1}{5}\begin{pmatrix}34 & -12\\ -12 & 41\end{pmatrix}=\frac{34+41}{5\cdot 2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{34-41}{5\cdot 2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\frac{-12}{5}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+0\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$=\frac{75}{5\cdot 2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{-7}{5\cdot 2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\frac{-12}{5}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

H-*eigenvalues:* $\begin{cases}\omega_\uparrow=\frac{75}{10}+\frac{5}{2}=10\\ \omega_\downarrow=\frac{75}{10}-\frac{5}{2}=5\end{cases}$

This gives: $\Omega_0=\frac{75}{10}$ and $\frac{1}{2}\hat{\boldsymbol{\Omega}}=\frac{1}{2}(\Omega_A,\Omega_B,\Omega_C)=(\frac{-7}{10},\frac{-24}{10},0)$ and $\frac{1}{2}|\boldsymbol{\Omega}|=\frac{1}{2}\sqrt{\Omega_A^2+\Omega_B^2+\Omega_C^2}=\frac{25}{10}=\frac{5}{2}$, which relates to eigenfrequencies found above: $\omega_\uparrow=\Omega_0+\frac{1}{2}|\Omega|=\frac{75}{10}+\frac{5}{2}=10$ and $\omega_\downarrow=\Omega_0-\frac{1}{2}|\Omega|=\frac{75}{10}-\frac{5}{2}=5$ and *squares* below.

$$\mathbf{K}=\begin{pmatrix}52 & -36\\ -36 & 73\end{pmatrix}=\frac{52+73}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{52-73}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-36\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+0\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

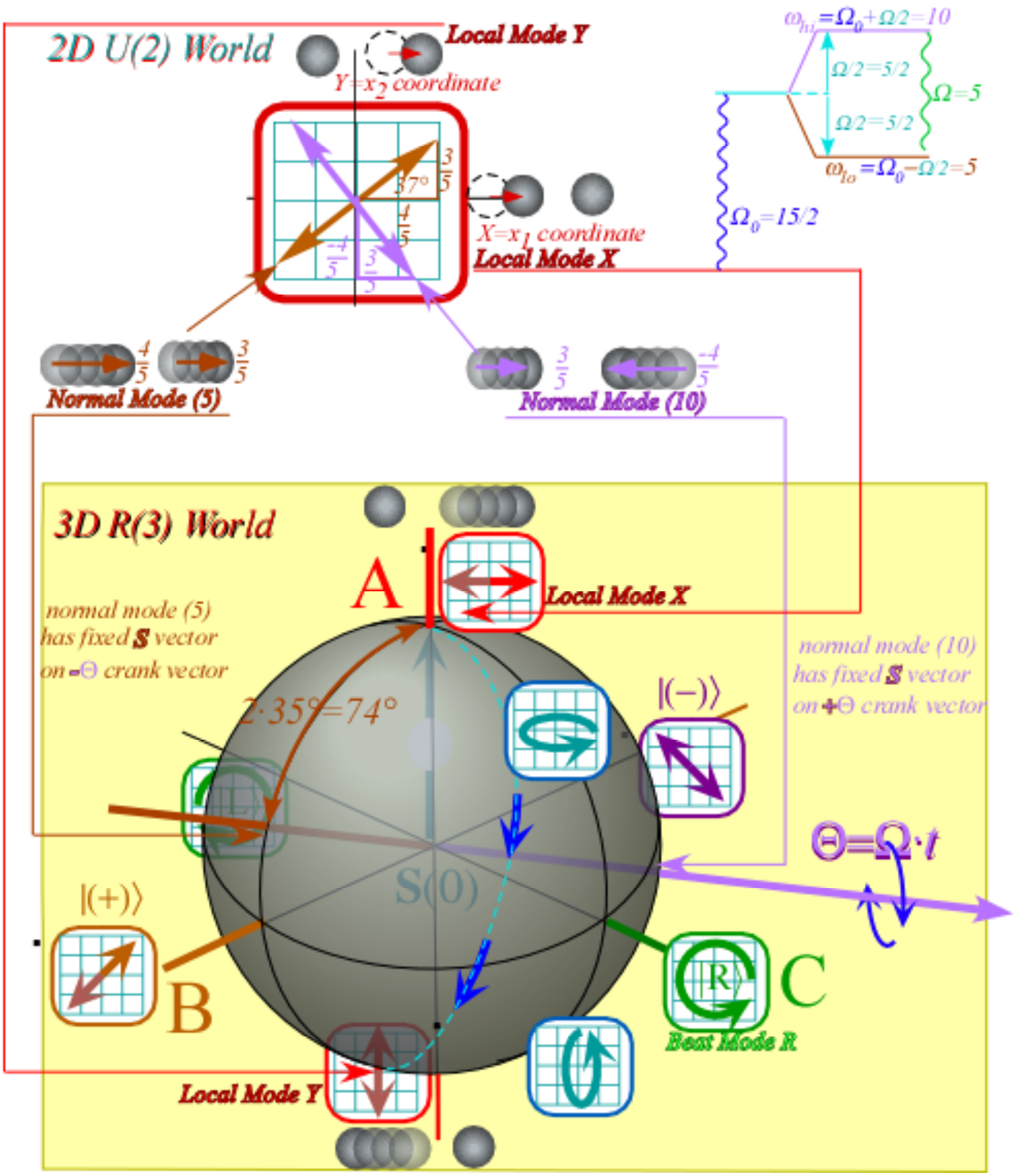
$$=\frac{125}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{-21}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-36\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

K-*eigenvalues:* $\begin{cases}K_\uparrow=\frac{125}{2}+\frac{75}{2}=100\\ K_\downarrow=\frac{125}{2}-\frac{75}{2}=25\end{cases}$

1(e) To show the (XY)-path is a parabola let $x=\cos\omega t$ so $y=-\cos 2\omega t=-\mathrm{Re}[e^{i2\omega t}]=-\mathrm{Re}[e^{i\omega t}]^2=-\mathrm{Re}[\cos\omega t+i\sin\omega t]^2$ gives:

$y=-\mathrm{Re}[\cos^2\omega t-\sin^2\omega t+i\ldots]=1-2\cos^2\omega t=1-2x^2$ {See (X1,X2) plot above left and below.}

Note that the 3D sketch above has C (Y axis) pointing up while the figure below has the A (or Z) axis pointing up. (Well, C rhymes with Z.)/s


2D U(2) World
Local Mode Y
Y=x2 coordinate
X=x1 coordinate
Local Mode X
37°
Normal Mode (5)
Normal Mode (10)
ωhi=Ω0+Ω/2=10
Ω/2=5/2
Ω/2=5/2
Ω=5
ωlo=Ω0−Ω/2=5
Ω0=15/2
3D R(3) World
A
Local Mode X
normal mode (5)
has fixed S vector
on −Θ crank vector
2·35°=74°
normal mode (10)
has fixed S vector
on +Θ crank vector
|(−)⟩
|(+)⟩
Θ=Ω·t
S(0)
B
C
Beat Mode R
Local Mode Y

Solutions for driver matrices in **Ex.2(a) to (e)**.:

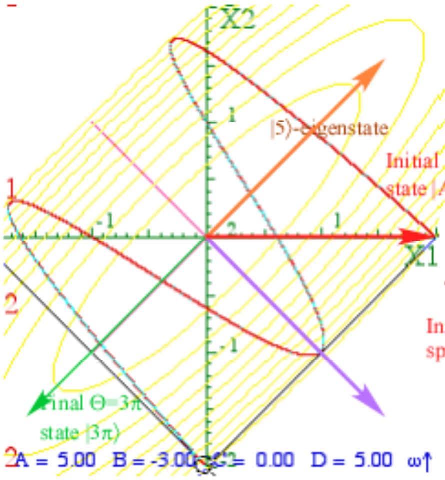
$$\mathbf{H}=\begin{pmatrix}5 & -3\\ -3 & 5\end{pmatrix}=5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-6\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$$

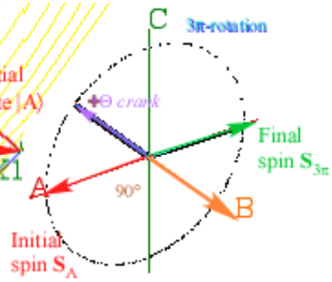
$$=\mathbf{51}-6\mathbf{S}_B=\mathbf{51}-3\boldsymbol{\sigma}_B$$

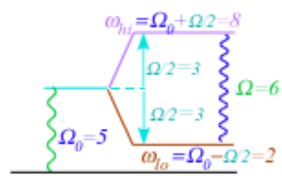
$$=\mathbf{51}+\Omega_B\mathbf{S}_B\ \therefore\left[\vec{\boldsymbol{\Omega}}=(\Omega_A,\Omega_B,\Omega_C)=(0,-6,0)\right]$$

$$\mathbf{H}^2=\mathbf{K}=\begin{pmatrix}34 & -30\\ -30 & 34\end{pmatrix}=\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}$$

$\Omega_0=5$, $\frac{1}{2}|\boldsymbol{\Omega}|=3$, $\Omega_0+\frac{1}{2}|\boldsymbol{\Omega}|=5+3=8$, $\Omega_0-\frac{1}{2}|\boldsymbol{\Omega}|=5\text{-}3=2$.







To find the (XY)-path let $x=\cos\omega t$ so $y=-\cos 4\omega t=-\mathrm{Re}[e^{i4\omega t}]=-\mathrm{Re}[e^{i\omega t}]^4=-\mathrm{Re}[\cos\omega t+i\sin\omega t]^4$ gives:

$$y=-\mathrm{Re}[\cos^4\omega t-6\cos^2\omega t\,\sin^2\omega t+\sin^4\omega t+i\ldots]$$

$$y=-\mathrm{Re}[x^4-6x^2y^2+y^4+i\ldots]=-\mathrm{Re}[x^4-6x^2(1-x^2)+(1-x^2)^2+i\ldots]$$

$$y=-1+8x^2-8x^4$$

The functions resulting from 4th harmonic is the 4th degree *Tschebycheff polynomial T_4*.

The n^{th} harmonic is the *n^{th} degree Tschebycheff T_n*.

Solutions for projectors in **Ex.3)**.:

The H matrix being solved on p. 60 to 65 of Lect. 23 is

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3} - i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3} + i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$

eigenvalue − 1

$$\omega_{\uparrow} = 10 + \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2} = 10 + 4 = 14$$

eigenvalue − 2

$$\omega_{\downarrow} = 10 - \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2} = 10 - 4 = 6$$

eigenvector − 1

$$\left|\uparrow\right\rangle = \begin{pmatrix} e^{-i\frac{\pi}{8}}\cos\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\sin\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \end{pmatrix} \frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

eigenvector − 2

$$\left|\downarrow\right\rangle = \begin{pmatrix} -e^{-i\frac{\pi}{8}}\sin\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} -e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \\ 1 \end{pmatrix} \frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

Using eigenvalues 14 and 6 we insert them into matrix H to make projectors:

$$\mathbf{P}_{14} = \frac{\begin{pmatrix} 12-6 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8-6 \end{pmatrix}}{14-6=8} = \frac{1}{8}\begin{pmatrix} 6 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 2 \end{pmatrix} \qquad \mathbf{P}_{6} = \frac{\begin{pmatrix} 12-14 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8-14 \end{pmatrix}}{6-14=-8} = \frac{1}{8}\begin{pmatrix} 2 & -\sqrt{6}(1-i) \\ -\sqrt{6}(1+i) & 6 \end{pmatrix}$$

Eigenvectors match up to an overall phase and norm that would be determined by physical considerations.

Exercises 4

4**(a)** Uncoupled oscillation of the AD type (p.71). Let Hamiltonian have A=1.5 and D=0.5 with B=0=C with initial oscillator variables $x_1(0)=1=x_2(0)$ and $p_1(0)=0=p_2(0)$. Describe initial spin vector (p.80 Lect.22) and its evolution.

$$\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \begin{pmatrix} 1+i0 \\ 1+i0 \end{pmatrix} = A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \sqrt{2}\begin{pmatrix} e^{-i\frac{0}{2}}\cos\frac{\frac{\pi}{2}}{2} \\ e^{i\frac{0}{2}}\sin\frac{\frac{\pi}{2}}{2} \end{pmatrix}, \quad \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D=1 \\ 2B=0 \\ 2C=0 \end{pmatrix} \text{ and: } \Omega_0 = \frac{A+D}{2} = 1$$

Initial state has α=0=γ and β=π/2 so S-vector is along B. (Below) Crank vector is along A-axis.

Asymmetry $S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[x_1^2+p_1^2-x_2^2-p_2^2\right] = 0 = \frac{I}{2}\cos\beta$

Balance $S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \left[p_1p_2+x_1x_2\right] = 1 = \frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \left[x_1p_2-x_2p_1\right] = 0 = \frac{I}{2}\sin\alpha\sin\beta$

Crank **Ω**=(1,0,0) rotates initial B-spin vector **S**=(S_A,S_B,S_C)=(0,1,0) around A-axis thru -C, -B, and +C axes.

Exercise 5 Given crank 4-vector: $\begin{pmatrix} \Omega_0 & \Omega_A & \Omega_B & \Omega_C \end{pmatrix} = \begin{pmatrix} \frac{A+D}{2} & \frac{A-D}{2} & B & C \end{pmatrix}$

We need crank to be on the cubic (1,1,1)-diagonal $\Omega_A = \Omega_B = \Omega_C$ so we set $\Omega_A = \Omega_B = B = \Omega_C = C = 0.1$

Solving for A and D: $A+D = 2\Omega_0$ or: $A = \Omega_0 + \Omega_A = 4 + 0.1 = 4.1$
$A-D = 2\Omega_A$ $D = \Omega_0 - \Omega_A = 4 - 0.1 = 3.9$

Trjectory screen shot below: