Assignment 10 - **PHYS 5103**-11/06/19-Due Wed. Nov. 13 CMwBang! Ch 4.1 thru Ch.4.4. and Lectures 20-21

Ex.1 *The "standard" Lorentzian (Note: Review complex 2-pole potential $\phi(z)=1/z$ and $f(z)=-1/z^2$ (10.42) in Unit 1-Ch.10 Fig.10.11.)*

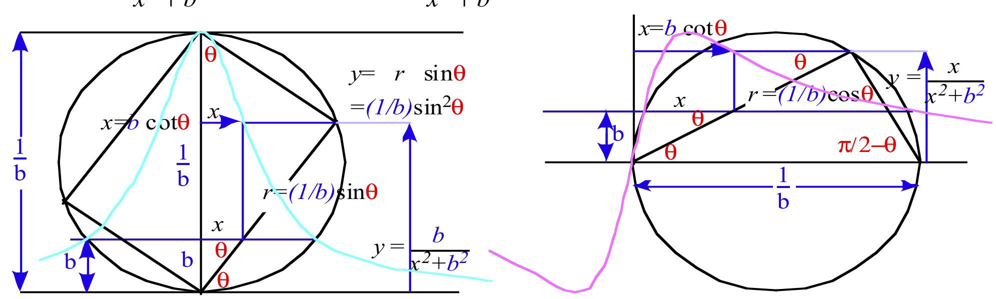
In physics literature, a standard Lorentzian function generally means a form $\mathrm{Im}\,L(\Delta)=\Gamma/(\Delta^2+\Gamma^2)$ with constant Γ. In the *Near-Resonant Approximation* (NRA is (4.2.18) and (4.2.33)) the $L(\Delta)$ is a low Δ and Γ approximation to exact G-equations (4.2.15). A clear NRA derivation is given in Lect. 20 p. 49 to 53 and geometries of these NRA are sketched on p. 58 to 68.

(a) Reduce (4.2.15) to NRA $L(\Delta-i\Gamma)=\mathrm{Re}\,L+i\,\mathrm{Im}\,L=|L|e^{i\rho}$ functions of detuning "beat rate" $\Delta=\omega_s-\omega_0$, decay rate Γ, and phase lag angle ρ. Indicate what part of these expressions is the standard Lorentzian.

(b) Show that NRA for complex response $G=\mathrm{Re}\,G+i\mathrm{Im}\,G$ gives circular arcs in the complex $\omega=|\omega|e^{i\theta}=|\omega|e^{i\rho}=\Delta+i\Gamma$ plane for constant decay rate Γ and variable detuning or beat rate Δ. How does this circle deviate from what is almost a circle in Fig. 4.2.6? (Consider higher Γ values for which NRA breaks down such as Fig. 4.2.14.) Relate to dipole scalar-Φ and vector-A potential field values plotted over coordinate lines for dipole force function $f(z)=1/z^2$ discussed in Ch. 10 of Unit 1. (See (10.42) and Fig. 10.11.)

(c) Do ruler-&-compass construction of NRA versions of the following Lorentz functions in figures below for $b=\frac{1}{2}$ and for $b=\frac{1}{4}$. Construction is similar to that of IHO elliptical orbits (Unit 1 Fig. 3.6 p. 53 or Lect.7 p.22) in that it involves 90° points of a zig-zags.

$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{x}{x^2+b^2} \text{ and } \mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{b}{x^2+b^2}$$.(See p. 58-68 of Lect. 20.)



(d) (*Xtra credit*)Study the Riemann-Cauchy equations for analytic function G* of Δ-iΓ that relate Δ and Γ partial derivatives of G^*_{Re} and G^*_{Im} (Recall Unit 1 eq.(10.32) or (better) Lect. 12 p.61) and consider what max our min values result from those derivatives being zero.

Ex.2 *Max and min G-values (Part (b-c) involves some derivative algebra!)*

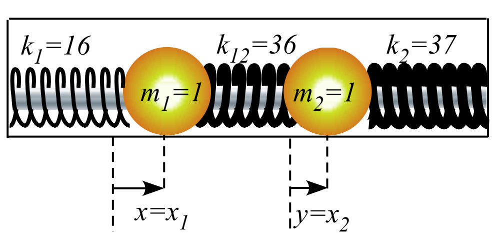
Derive equations for the extreme values for the *exact* Lorentz-Green response functions $G_{\omega_0}(\omega_s)$ as asked below.

Compare these to *Near-Resonant Approximations* (*NRA*) given in preceding **Ex.1**.Exact plots by calculator help to check algebraic answers.

(a₁) Find values which give maxima for: $\mathrm{Re}\,G_{\omega_0}(\omega_s)$, $\mathrm{Im}\,G_{\omega_0}(\omega_s)$, and $|G_{\omega_0}(\omega_s)|$ assuming ω_0 is constant and ω_s varies.

(a₂) Find values which give maxima for: $\mathrm{Re}\,G_{\omega_0}(\omega_s)$, $\mathrm{Im}\,G_{\omega_0}(\omega_s)$, and $|G_{\omega_0}(\omega_s)|$ assuming ω_s is constant and ω_0 varies.

Do (**a₁**) and (**a₂**) give the same results?



Ex.3 *Coupled oscillation by projection* **P**-*operators*

Two identical mass $M=1kg$ blocks slide friction-free on a rod and are connected by springs $k_1=16N\cdot m^{-1}$ and $k_2=37N\cdot m^{-1}$ to ends of a box and coupled to each other by spring $k_{12}=36N\cdot m^{-1}$.

(a) Write Lagrangian equations of motion and derive a **K**-matrix form of them.

(b) Solve for eigenmodes and eigenfrequencies of system and plot their directions on an X,Y-graph. Use spectral decomposition methods (Lect. 21 p. 36-53 or Appendix 4.C) to derive eigensolution projectors and eigenvectors.

(c) Given initial conditions ($X(0)=1, Y(0)=0, \mathbf{V}_0=0$), plot the resulting path in the XY-plane. Show it is a parabola.(*Tschebycheff* function)

(d) Use spectral decomposition (Lect. 21 or Appendix 4.C) to derive square-roots **H**=√**K**. (How many different square-roots does **K** have?) (This is an important part of relating *Classical* coupled oscillators to *Quantum* coupled oscillators. See Lect. 22.)

Assignment 10 ***Solutions to Oscillator Response*** **Ex.1** *and* **Ex.2**

Near-Resonant-Approximate (NRA) Lorentz functions $G=1/H$ solve 1st order equations of form $H\cdot\psi=\phi$. That is, $\psi=(1/H)\cdot\phi=G\cdot\phi$, where $H=\Delta+i\Gamma$ or $H^*=\Delta-i\Gamma$ and $G=1/(\Delta+i\Gamma)$ or $G^*=1/(\Delta-i\Gamma)$ are given below.

$$G=\frac{1}{\Delta+i\Gamma}=\frac{1}{\Delta+i\Gamma}\frac{\Delta-i\Gamma}{\Delta-i\Gamma}=\frac{\Delta}{\Delta^2+\Gamma^2}+i\frac{-\Gamma}{\Delta^2+\Gamma^2}\text{ and conjugate: }\quad G^*=\frac{1}{\Delta-i\Gamma}=\frac{\Delta}{\Delta^2+\Gamma^2}+i\frac{\Gamma}{\Delta^2+\Gamma^2}$$

This uses the complex inversion function $f(z)=1/z$ and (more commonly) its conjugate $f^*(z)=f(z^*)=1/z^*$.

$$z=x+iy=re^{+i\theta}\qquad x=\frac{z+z^*}{2}\qquad \frac{1}{z}=\frac{x-iy}{x^2+y^2}=\frac{z^*}{r^2}=\frac{e^{-i\theta}}{r}$$

$$z^*=x-iy=re^{-i\theta}\qquad y=\frac{z-z^*}{2i}\qquad \frac{1}{z^*}=\frac{x+iy}{x^2+y^2}=\frac{z}{r^2}=\frac{e^{+i\theta}}{r}$$

The fact that the complex z-derivative of $f(z^*)$ is identically zero gives real derivative chain-relations.

$$0=\frac{df(z^*)}{dz}=\frac{\partial x}{\partial z}\frac{\partial f^*}{\partial x}+\frac{\partial y}{\partial z}\frac{\partial f^*}{\partial y}=\frac{1}{2}\frac{\partial f^*}{\partial x}+\frac{1}{2i}\frac{\partial f^*}{\partial y}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(f_x^*+if_y^*)=\frac{1}{2}\left(\frac{\partial f_x^*}{\partial x}+\frac{\partial f_y^*}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial f_x^*}{\partial y}-\frac{\partial f_y^*}{\partial x}\right)$$

$$\text{Let: }\mathrm{Re}f(z^*)=f_x^*\qquad\text{and: }\mathrm{Im}f(z^*)=f_y^*\qquad =\frac{1}{2}\left(\nabla\bullet\mathbf{f}^*\right)\quad+\frac{i}{2}\left(\nabla\times\mathbf{f}^*\right)$$

The zeroing of $\nabla\bullet\mathbf{f}^*$ and $\nabla\times\mathbf{f}^*$ are called *Riemann-Cauchy relations*. For Lorentz function G^* we have:

$$\frac{\partial G^*_{\mathrm{Re}}}{\partial\Delta}=\frac{\partial}{\partial\Delta}\frac{\Delta}{\Delta^2+\Gamma^2}=-\frac{\partial}{\partial\Gamma}\frac{\Gamma}{\Delta^2+\Gamma^2}=-\frac{\partial G^*_{\mathrm{Im}}}{\partial\Gamma}\qquad \frac{\partial G^*_{\mathrm{Re}}}{\partial\Gamma}=\frac{\partial}{\partial\Gamma}\frac{\Delta}{\Delta^2+\Gamma^2}=\frac{\partial}{\partial\Delta}\frac{\Gamma}{\Delta^2+\Gamma^2}=\frac{\partial G^*_{\mathrm{Im}}}{\partial\Delta}$$

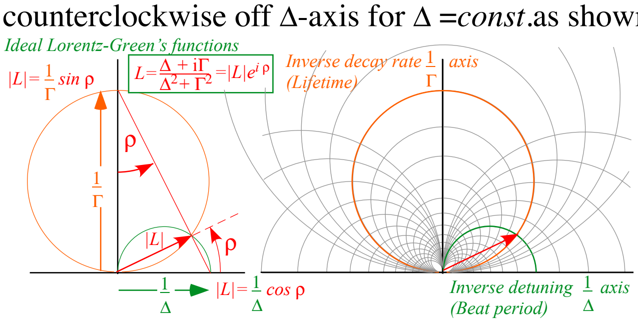
$$=-\frac{\Delta^2-\Gamma^2}{(\Delta^2+\Gamma^2)^2}\qquad\qquad =\frac{-2\Delta\Gamma}{(\Delta^2+\Gamma^2)^2}$$

Zero Δ-derivative of G^*_{Re} and max/min $G^*_{\mathrm{Re}}=1/2\Delta$ occur if $\Delta=\pm\Gamma$, when $G^*_{\mathrm{Im}}=1/2\Gamma$ is half its max value $1/\Gamma$. Zero Δ-derivative of G^*_{Im} and max/min $G^*_{\mathrm{Im}}=1/\Gamma$ occur when $\Delta=0$, where $G^*_{\mathrm{Re}}=0$ is zero.

There is symmetry between these functions. Just flip Δ with Γ and G^*_{Re} with G^*_{Im} and get:

Zero Γ-derivative of G^*_{Im} and max/min $G^*_{Im}=1/2\Gamma$ occur if $\Gamma=\pm\Delta$, when $G^*_{\mathrm{Re}}=1/2\Delta$ is half its max value $1/\Delta$. Zero Γ-derivative of G^*_{Re} and max/min $G^*_{\mathrm{Re}}=1/\Delta$ occur when $\Gamma=0$, where $G^*_{\mathrm{Im}}=0$ is zero.

G functions describe orthogonal circles using angle $\theta=\rho$ measured clockwise off Γ-axis for $\Gamma=const$. or measured counterclockwise off Δ-axis for $\Delta=const$.as shown in the figure 4.2.13. (Below)



(From Fig. 4.2.13)

Fig. 4.2.14 shows non-ideal Lorentzian geometry with asymmetric circles as described in **Ex.2**.

Assignment 10 ***Solutions to Oscillator Response problems*** **Ex.2**

Extrema for X=Re$G_\Gamma(\omega,\omega_0)$, Im$G_\Gamma(\omega,\omega_0)$, $|G_\Gamma(\omega,\omega_0)|$ defined by $\frac{\partial X}{\partial \omega_0}=0$ may differ from ones with $\frac{\partial X}{\partial \omega}=0$.

$$G_\Gamma(\omega_0,\omega) = \mathrm{Re}\,G_\Gamma(\omega_0,\omega) + i\,\mathrm{Im}\,G_\Gamma(\omega_0,\omega) = |G_\Gamma(\omega_0,\omega)|\, e^{i\rho}$$

$$\frac{1}{\omega_0^2-\omega^2-i2\Gamma\omega} = \frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2} + i\frac{2\Gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2} = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}}e^{i\rho}$$

Real G *versus* stimulus frequency ω:

$$0=\frac{\partial}{\partial\omega}\mathrm{Re}\,G_\Gamma(\omega_0,\omega)=\frac{-2\omega}{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}-\frac{(\omega_0^2-\omega^2)\frac{\partial}{\partial\omega}[\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2]}{[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]^2}$$

$$0=-2\omega[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]-(\omega_0^2-\omega^2)[2(\omega_0^2-\omega^2)(-2\omega)+2(2\Gamma\omega)2\Gamma]$$

$$0=-2\omega[(\omega_0^2-\omega^2)^2+4\Gamma^2\omega^2]+2\omega(\omega_0^2-\omega^2)2(\omega_0^2-\omega^2)-2\omega(\omega_0^2-\omega^2)4\Gamma^2$$

$$0=-(\omega_0^2-\omega^2)^2-4\Gamma^2\omega^2+2(\omega_0^2-\omega^2)^2-(\omega_0^2-\omega^2)4\Gamma^2$$

$$0=(\omega_0^2-\omega^2)^2-\omega_0^2 4\Gamma^2=\omega^4-2\omega_0^2\omega^2+\omega_0^4-\omega_0^2 4\Gamma^2 \quad \text{has solutions: } \omega = \sqrt{\omega_0^2\pm 2\omega_0\Gamma}\cong\omega_0\pm\Gamma\ldots$$

Real G *versus* oscillator natural frequency ω_0:

$$0=\frac{\partial}{\partial\omega_0}\mathrm{Re}\,G_\Gamma(\omega_0,\omega)=\frac{2\omega_0}{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}-\frac{(\omega_0^2-\omega^2)\frac{\partial}{\partial\omega_0}[\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2]}{[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]^2}$$

$$0=-2\omega_0[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]-(\omega_0^2-\omega^2)[2(\omega_0^2-\omega^2)2\omega_0]=-(\omega_0^2-\omega^2)^2+4\Gamma^2\omega^2$$

$$0=\omega_0^4-(2\omega_0^2+4\Gamma^2)\omega^2+\omega^4 \quad \text{has solutions: } \omega_0=\sqrt{\omega^2\pm 2\omega\Gamma}\cong\omega\pm\Gamma$$

Imaginary G *versus* stimulus frequency ω:

$$0=\frac{\partial}{\partial\omega}\mathrm{Im}\,G_\Gamma(\omega_0,\omega)=\frac{2\Gamma}{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}-\frac{2\Gamma\omega\frac{\partial}{\partial\omega}[\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2]}{[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]^2}$$

$$0=2\Gamma[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]-2\Gamma\omega[2(\omega_0^2-\omega^2)(-2\omega)+2(2\Gamma\omega)2\Gamma]$$

$$0=3\omega^4-(2\omega_0^2-4\Gamma^2)\omega^2-\omega_0^4 \quad \text{has solutions: } \omega=\sqrt{\frac{\omega_0^2-2\Gamma^2\pm 2\sqrt{\omega_0^2(\omega_0^2-\Gamma^2)+\Gamma^4}}{3}}.$$

Imaginary G *versus* oscillator natural frequency ω_0:

$$0=\frac{\partial}{\partial\omega_0}\mathrm{Im}\,G_\Gamma(\omega_0,\omega)=-\frac{2\Gamma\omega\frac{\partial}{\partial\omega_0}[\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2]}{[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]^2}$$

$$0=-2\Gamma\omega[2(\omega_0^2-\omega^2)2\omega_0 \quad \text{has solutions: } \omega_0=\omega$$

Magnitude $|G|$ *versus* stimulus frequency ω:

$$0=\frac{\partial}{\partial\omega}|G_\Gamma(\omega_0,\omega)|=\frac{\partial}{\partial\omega}\frac{1}{[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]^{1/2}}=-\frac{1}{2}\frac{\frac{\partial}{\partial\omega}[\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2]}{[(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2]^{3/2}}$$

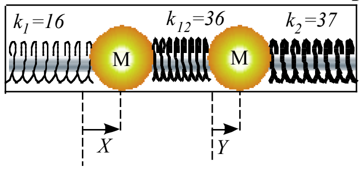
$$0=2(\omega_0^2-\omega^2)2\omega+2(2\Gamma\omega)2\Gamma] \text{ or: } 0=\omega_0^2-\omega^2+2\Gamma^2 \text{ has solutions: } \omega=\sqrt{\omega_0^2-2\Gamma^2}\cong\omega_0-\frac{\Gamma^2}{\omega_0}\ldots$$

Interesting case (not assigned) E~ $m\omega^2|G|^2$ *versus* stimulus frequency ω:

$$0=\frac{\partial}{\partial\omega}|\omega G_\Gamma(\omega_0,\omega)|^2=\frac{\partial}{\partial\omega}\frac{\omega^2}{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}=\frac{2\omega}{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}-\frac{\omega^2\cdot\frac{\partial}{\partial\omega}[\left(\omega_0^2-\omega^2\right)^2+\left(2\Gamma\omega\right)^2]}{(\omega_0^2-\omega^2)^2+(2\Gamma\omega)^2}$$

$$0=-2(\omega_0^2-\omega^2)^2\omega^2 \quad \text{has solutions: } \omega=\omega_0$$

Assignment 10 – Solutions to **Ex.3**: *Normal mode and coupled oscillators*



Lagrangian: $L = T - V = \frac{1}{2}(M\dot{X}^2 + M\dot{Y}^2) - \frac{1}{2}(k_1X^2 + k_2Y^2 + k_{12}(X-Y)^2)$ Hamiltonian: $H=T+V$.

$$= \frac{1}{2}\dot{\mathbf{X}}\bullet\begin{pmatrix} M & 0 \\ 0 & M\end{pmatrix}\bullet\dot{\mathbf{X}} - \frac{1}{2}\mathbf{X}\bullet\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}\bullet\mathbf{X}$$

gives: $M\ddot{\mathbf{X}} = -\mathbf{K}\cdot\mathbf{X}$

Find eigenbase vectors that diagonalize $\mathbf{K} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix} = \begin{pmatrix} 16+36 & -36 \\ -36 & 37+36\end{pmatrix} = \begin{pmatrix} 52 & -36 \\ -36 & 73\end{pmatrix}$

Eigenvalue secular equation: $\lambda^2 - (trace\mathbf{K})\lambda + (\det\mathbf{K}) = 0 = \lambda^2 - (125)\lambda + 2500 = (\lambda - 25)(\lambda - 100)$

E-vector projectors: $\mathbf{P}_{25} = \frac{\mathbf{K} - 100\cdot\mathbf{1}}{25-100} = \frac{1}{-75}\begin{pmatrix} 52-100 & -36 \\ -36 & 73-100\end{pmatrix}$, $\mathbf{P}_{100} = \frac{\mathbf{K} - 25\cdot\mathbf{1}}{100-25} = \frac{1}{75}\begin{pmatrix} 52-25 & -36 \\ -36 & 73-25\end{pmatrix}$

$$\mathbf{P}_{25} = \frac{1}{75}\begin{pmatrix} 48 & 36 \\ 36 & 27\end{pmatrix} = \frac{1}{25}\begin{pmatrix} 16 & 12 \\ 12 & 9\end{pmatrix} \qquad \mathbf{P}_{100} = \frac{1}{75}\begin{pmatrix} 27 & -36 \\ -36 & 48\end{pmatrix} = \frac{1}{25}\begin{pmatrix} 9 & -12 \\ -12 & 16\end{pmatrix}$$

Lo-**K**-eigenvalue: $\lambda_\downarrow=25$ eigenfrequency: $\omega_\downarrow=\sqrt{\lambda}=5$. Hi-**K**-eigenvalue: $\lambda_\uparrow=100$ eigenfrequency: $\omega_\uparrow=\sqrt{\lambda}=10$.

K-e-vectors: $\left|25\right\rangle = \frac{1}{25}\begin{pmatrix} 12 \\ 9\end{pmatrix}\cdot\frac{1}{norm} = \frac{1}{25}\begin{pmatrix} 12 \\ 9\end{pmatrix}\cdot\frac{1}{\sqrt{\frac{9}{25}}} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5}\end{pmatrix}$ $\left|100\right\rangle = \frac{1}{25}\begin{pmatrix} 9 \\ -12\end{pmatrix}\cdot\frac{1}{norm} = \frac{1}{25}\begin{pmatrix} 9 \\ -12\end{pmatrix}\cdot\frac{1}{\sqrt{\frac{9}{25}}} = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5}\end{pmatrix}$

(e-vector norm lies on **P**-diagonal of chosen column, here 2nd column of $\mathbf{P}_{25}$ and 1st column of $\mathbf{P}_{100}$.)

Decomposition of **K**: $\mathbf{K} = \begin{pmatrix} 52 & -36 \\ -36 & 73\end{pmatrix} = 25\mathbf{P}_{25} + 100\mathbf{P}_{100}$

Square root of **K**: $\sqrt{\mathbf{K}} = \sqrt{\begin{pmatrix} 52 & -36 \\ -36 & 73\end{pmatrix}} = \sqrt{25}\mathbf{P}_{25} + \sqrt{100}\mathbf{P}_{100}$

$$= \frac{1}{5}\begin{pmatrix} 16 & 12 \\ 12 & 9\end{pmatrix} + \frac{2}{5}\begin{pmatrix} 9 & -12 \\ -12 & 16\end{pmatrix} = \frac{1}{5}\begin{pmatrix} 34 & -12 \\ -12 & 41\end{pmatrix} = \mathbf{H} \qquad \frac{1}{5}\begin{pmatrix} -34 & 12 \\ 12 & -41\end{pmatrix}$$

$= \frac{1}{5}\begin{pmatrix} 16 & 12 \\ 12 & 9\end{pmatrix} - \frac{2}{5}\begin{pmatrix} 9 & -12 \\ -12 & 16\end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & 36 \\ 36 & -23\end{pmatrix}$ ...the 2nd (of 4) square roots $\frac{1}{5}\begin{pmatrix} 2 & -36 \\ -36 & 23\end{pmatrix}$ (the other 2)

Note also that square of general *(C=0)*-matrix **H** is $\mathbf{K}=\mathbf{H}^2 = \begin{pmatrix} A & B \\ B & D\end{pmatrix}\begin{pmatrix} A & B \\ B & D\end{pmatrix} = \begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & D^2+B^2\end{pmatrix}$.

This will be an important part of relating *Classical* coupled oscillators to *Quantum* coupled oscillators and two-level systems like spin-½ . (Next Lecture 22.)